

Maslov indices for periodic orbits

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Abstract

It is shown that there is a generalization of the Conley-Zehnder index for periodic trajectories of a classical Hamiltonian system (Q, ω, H) from the case $Q = T^*\mathbf{R}^n$ to arbitrary symplectic manifolds. As it turns out, it is precisely this index which appears as a ‘Maslov phase’ in the trace formulas by Gutzwiller and Duistermaat-Guillemin.

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0 Introduction

Consider a classical mechanical system (Q, ω, H) , where Q is a symplectic manifold, ω its symplectic 2-form and $H \in C^\infty(Q, \mathbf{R})$ a Hamiltonian. Assume that $\gamma : \mathbf{R} \rightarrow Q$ is a periodic solution curve of period T and energy E for the Hamiltonian vector field X_H on Q . The closed trajectory is called *regular* if the (linear) Poincaré map $P(T)$ has no unit eigenvalue. From the implicit function theorem it follows that regular periodic trajectory always come in 1-parameter families. Their ‘orbit cylinder’ is a symplectic submanifold of Q which is transversal to all energy surfaces $H^{-1}(E)$. We refer to [1] for a detailed proof, which also shows that orbit cylinders are stable under small perturbations of the Hamiltonian.

Conley and Zehnder [2] have defined an index $\text{ind}_{CZ}(\gamma)$ for regular periodic orbits in $T^*\mathbf{R}^n$, generalizing the usual Morse index for closed geodesics on a Riemannian manifold. Roughly speaking, the index measures how often neighbouring trajectories of the same energy wind round the orbit. It is stable under deformations of the orbit as long as the regularity assumption is not violated. In particular, all members of the orbit cylinder have the same index.

As we will see below, the Conley-Zehnder index admits a natural extension to arbitrary symplectic manifolds. The construction will only depend on the choice of a homotopy class of Lagrangian subbundles L of TQ along the orbit $\gamma^\# = \gamma(\mathbf{R})$. Such a choice is often dictated by the particular system under study, and is natural e.g. for cotangent bundles $Q = T^*X$ or if the orbit is contractible. The index $\text{ind}(\gamma, L)$ is characterized by the following two properties:

- I1 The index is stable under small perturbations of the system (as long as the orbit remains regular).
- I2 Whenever there is an invariant Lagrangian subbundle M of TQ along the orbit, $\text{ind}(\gamma, L)$ is the (Maslov) intersection number of M with L .

We show in [8] that it is precisely this index that appears as a Maslov phase in the ‘trace formulas’ by Gutzwiller [5] and Duistermaat-Guillemin [4]. For closely related results, see Duistermaat [3] and Robbins [9].

1 The Intersection Number of Lagrangian Subspaces

Let (E, ω) be a real symplectic vector space of dimension $2n$ and let $\Lambda(E)$ be its Lagrangian Grassmannian, i.e. the set of Lagrange subspaces of E . Consider the action of the symplectic group $\text{Sp}(E)$ on the set $\Lambda(E)^3$ of ordered Lagrangian triplets (L_1, L_2, L_3) . It is clear that the dimensions of the intersections are invariant under this action. Another independent invariant is the so-called *signature of a Lagrangian triplet* discovered by Hörmander

and Kashiwara [7]. Together, these invariants completely specify the relative position of three Lagrangian subspaces up to symplectic transformations. For $(L_1, L_2, L_3) \in \Lambda(E)^3$, the signature $s(L_1, L_2, L_3) \in \mathbf{Z}$ is defined as the signature of the quadratic form

$$Q(L_1, L_2, L_3) : L_1 \oplus L_2 \oplus L_3 \rightarrow \mathbf{R}, \quad (x_1, x_2, x_3) \mapsto \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1). \quad (1)$$

It is immediate from the definition that the signature $s : \Lambda(E)^3 \rightarrow \mathbf{Z}$ is invariant under symplectic transformations and antisymmetric under exchange of two of the L 's. Let us list some less trivial properties [6, 7]:

Proposition 1 1. $s(L_2, L_3, L_4) - s(L_1, L_3, L_4) + s(L_1, L_2, L_4) - s(L_1, L_2, L_3) = 0$.

2. *Reduction lemma: For arbitrary subspaces K of $L_1 \cap L_2 + L_2 \cap L_3 + L_3 \cap L_1$,*

$$s(L_1, L_2, L_3) = s(L_1^K, L_2^K, L_3^K), \quad (2)$$

where L_i^K denotes the image of L_i under the symplectic reduction $(K + K^\omega) \rightarrow E^K := (K + K^\omega)/(K \cap K^\omega)$.

3. *The signature runs through all integers between $-\frac{1}{2} \dim E^F$ and $+\frac{1}{2} \dim E^F$, where $F = (L_1 \cap L_2) + (L_2 \cap L_3) + (L_3 \cap L_1)$. Consequently, $s(L_1, L_2, L_3) + \dim(L_1 \cap L_2) + \dim(L_2 \cap L_3) + \dim(L_3 \cap L_1) + n$ is an even number.*

4. *The orbits of the action of $\mathrm{Sp}(E)$ on $\Lambda(E)^3$ are completely determined by $\dim(L_1 \cap L_2 \cap L_3)$, $\dim(L_1 \cap L_2)$, $\dim(L_2 \cap L_3)$, $\dim(L_3 \cap L_1)$ and $s(L_1, L_2, L_3)$: If these five numbers coincide for two triplets, they lie on the same orbit.*

5. *The signature is locally constant on the set of all triplets with given dimensions of intersections.*

Consider now two continuous paths $L_1, L_2 : [a, b] \rightarrow \Lambda(E)$, where $a \leq b$. Choose a sufficiently fine partition $a = t_0 \leq \dots \leq t_k = b$ and Lagrangian subspaces M_ν such that M_ν is transversal to all $L_i(t)$ with $t_{\nu-1} \leq t \leq t_\nu$, $i = 1, 2$. We define the *intersection number* $[L_1 : L_2]$ by

$$[L_1 : L_2] = \frac{1}{2} \sum_{\nu=1}^k \left(s(L_1(t_{\nu-1}), L_2(t_{\nu-1}), M_\nu) - s(L_1(t_\nu), L_2(t_\nu), M_\nu) \right). \quad (3)$$

Thanks to proposition 1, this definition is indeed independent of the specific choices. Since the expressions $s(L_1, L_2, M)$ are locally constant in M as long as $M \cap L_i = \{0\}$, the above definition also makes sense if one considers a symplectic vector bundle over $[a, b]$ and replaces the L_i by Lagrangian subbundles.

Proposition 2 (*Properties of the intersection number.*)

1. *Antisymmetry:* $[L_1 : L_2] + [L_2 : L_1] = 0$.

2. *Invariance:* $[A(L_1) : A(L_2)] = [L_1 : L_2]$ for all continuous paths $A : [a, b] \rightarrow \mathrm{Sp}(E)$.

3. $[L_1 : L_2] + \frac{1}{2} \dim(L_1(a) \cap L_2(a)) + \frac{1}{2} \dim(L_1(b) \cap L_2(b)) \in \mathbf{Z}$. *In particular, $[L_1 : L_2]$ is an integer if the intersections at the endpoints are transversal.*

4. $[L_1 : L_2] + [L_2 : L_3] + [L_3 : L_1] = \frac{1}{2} \left(s(L_1(a), L_2(a), L_3(a)) - s(L_1(b), L_2(b), L_3(b)) \right)$.

5. Consider the space of paths $L_1 \times L_2 : [a, b] \rightarrow \Lambda(E)^2$ with given dimensions of the intersections at the endpoints. $[L_1 : L_2]$ labels the connected components of this space.

Using the intersection number, one arrives at a straightforward construction of the so-called *Leray index* [6, 7]. Let $\pi : \tilde{\Lambda}(E) \rightarrow \Lambda(E)$ denote the universal covering of the Lagrange-Grassmann manifold. For $u_0, u_1 \in \tilde{\Lambda}(E)$, choose any path $L : [0, 1] \rightarrow \Lambda(E)$ such that $L(0) = \pi(u_0)$ and $L(1) = \pi(u_1)$. Define the Leray index $m(u_0, u_1) \in \frac{1}{2}\mathbf{Z}$ by

$$m(u_0, u_1) = [L(t) : L(1)] = [L(t) : L(0)].$$

Proposition 2 guarantees that this is independent of the chosen path and immediately leads to the following statements:

Proposition 3 (*Properties of Leray's index.*)

1. For $L_i = \pi(u_i)$, Leray's formula holds:

$$m(u_1, u_2) + m(u_2, u_3) + m(u_3, u_1) = \frac{1}{2}s(L_1, L_2, L_3). \quad (4)$$

2. For arbitrary lifts $u_i(\cdot)$ of Lagrangian curves $L_i(\cdot)$,

$$[L_1 : L_2] = m(u_1(a), u_2(a)) - m(u_1(b), u_2(b)). \quad (5)$$

3. $m(u_1, u_2)$ is locally constant on the set of all u_1, u_2 with fixed $\dim(L_1 \cap L_2)$.

Let $\tau : \widetilde{\mathrm{Sp}}(E) \rightarrow \mathrm{Sp}(E)$ denote the universal covering group of the symplectic group. Elements \tilde{A} of the covering group can be identified with homotopy classes of paths $A(t)$ in $\mathrm{Sp}(E)$ connecting the identity to $A = \tau(\tilde{A})$. Recall that the graph

$$\Gamma_B := \{(Bx, x) | x \in E\}$$

of a symplectic transformation B in E is a Lagrangian subspace of $E \times E^-$, which is $E \oplus E$ with the symplectic form $\mathrm{pr}_1^*\omega - \mathrm{pr}_2^*\omega$. We hence obtain an index

$$\mu : \widetilde{\mathrm{Sp}}(E) \rightarrow \frac{1}{2}\mathbf{Z}, \quad \tilde{A} \mapsto [\Delta : \Gamma_{A(t)}], \quad (6)$$

where Δ is the graph of the identity, i.e. the diagonal in $E \times E^-$. (Equivalent indices are introduced in [2] and [3].)

Proposition 4 (*Properties of the index μ .*)

1. $\mu(\tilde{A})$ is locally constant on the set of all \tilde{A} with given $\dim(\ker(A - I))$.
2. $\mu(\tilde{A}) + \frac{1}{2}\dim(\ker(A - I)) \in \mathbf{Z}$.
3. Let $A(\cdot) : [0, 1] \rightarrow \mathrm{Sp}(E)$ be any path representing \tilde{A} , and let $L, M \in \Lambda(E)$ be arbitrary. Then

$$\mu(\tilde{A}) = [M : A(t)L] + \frac{1}{2}s(\Delta, L \times M, \Gamma_A). \quad (7)$$

If $\ker(A - I)$ is symplectic and if L is A -invariant, the second term on the rhs vanishes.

4. (See ref. [2].) Two elements of the set of all \tilde{A} with $\ker(A - I) = \{0\}$ are in the same connected component if and only if they have the same index.

2 The index of periodic trajectories

Let us now return to the situation described in the introduction. It follows from the methods in [2] that there is at most one index $\text{ind}(\gamma, L)$ satisfying properties I1, I2. On the other hand, we can give an explicit expression for the index as follows.

Let $q \in \gamma^\#$. Let $\mathcal{E}_1 \subset T_{\gamma^\#}Q$ be the tangent bundle of the orbit cylinder and \mathcal{E}_2 its symplectic orthogonal. Choosing a symplectic trivialization of these bundles, we can consider the linearized flow as curves of linear symplectic transformations in $E_1 = \mathcal{E}_1(q)$ and $E_2 = \mathcal{E}_2(q)$ respectively. Write $P(t)$ (Poincare map) for the flow in E_2 , and choose any Lagrangian subspace M of $E_1 \oplus E_2$. Set

$$\text{ind}(\gamma, L) = [L(\gamma(t)) : M] + \mu(\widetilde{P(T)}). \quad (8)$$

It follows from the above propositions that this is well-defined, in particular it does not depend on the choice of the trivializations. There is an alternative formula for the index that does not require any such trivialization. Letting Γ_{TF^T} be the graph of the linearized flow, one has the following formula:

$$\text{ind}(\gamma, L) = [L(\gamma(t)) : TF^t(M)] + \frac{1}{2}s(\Delta, L_q \times M, \Gamma_{TF^T}) + \frac{1}{2}\text{sgn}\left(\frac{\partial T}{\partial E}\right). \quad (9)$$

where $M \in \Lambda(T_q Q)$ is arbitrary.

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